

SYMPLECTIC 4-MANIFOLDS WITH A FREE CIRCLE ACTION

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ABSTRACT. Let M be a symplectic 4-manifold admitting a free circle action. In this paper we show that, modulo suitable subgroup separability assumptions, the orbit space N admits a fibration over the circle. The separability assumptions are known to hold in several cases: in particular, this result covers the case where N has vanishing Thurston norm, or is a graph manifold. Furthermore, combining this result with the Lubotzky alternative, we show that if the symplectic structure has trivial canonical bundle then M is a torus bundle over a torus, confirming a folklore conjecture. We also generalize various constructions of symplectic structures on 4-manifold with a free circle action. The combination of our results allows us in particular to completely determine the symplectic cone of a 4-manifold with a free circle action such that the orbit space is a graph manifold.

1. INTRODUCTION AND MAIN RESULTS

The study of symplectic 4-manifolds, in spite of the enormous developments of the last 20 years, still faces the hurdle of answering one of the most basic questions, of smooth topology in character, namely determining which 4-manifolds admit a symplectic structure. Strengthened by the results of Taubes, that have allowed the use of Seiberg-Witten theory to address this question, a lot of interest has been devoted to what is perhaps the most elementary class of manifolds for which the question is non-obvious, namely 4-manifolds of the form $S^1 \times N$. Here, various authors have given convincing evidence to the conjecture that $S^1 \times N$ admits a symplectic structure if and only if N fibers over the circle. In [FV06b] the authors of the present paper proved the existence of a relation between this problem and the study of certain algebraic properties of the fundamental group of N . This relation allowed us to solve the conjecture in the affirmative for various classes of manifolds, and to relate the general solution to standard conjectures in 3-dimensional hyperbolic geometry.

In this paper we will extend our results in two directions, that correspond roughly to two parts. First, we will show how the techniques of [FV06b] can be applied to study the case of symplectic 4-manifolds M admitting a free circle action with non-trivial Euler class. Second, we will obtain new results on the existence of symplectic structures on 4-manifolds with a free circle action. Combining these two parts we get complete information on the symplectic cone for various classes of manifolds. (Recall that the symplectic cone of a 4-manifold M is the cone of elements of $H^2(M, \mathbb{R})$ that can be represented by symplectic form.)

Before stating our main results, we will introduce some notation. Given a 4-manifold M with a free circle action we denote the orbit space by N and we denote by $p : M \rightarrow N$ the

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quotient map, which defines a principal S^1 -bundle over N . We denote by $e \in H^2(N)$ the Euler class of the S^1 -bundle. Recall the Gysin sequence

$$(1) \quad \mathbb{Z} = H^0(N) \xrightarrow{\cdot e} H^2(N) \xrightarrow{p^*} H^2(M) \xrightarrow{p_*} H^1(N) \xrightarrow{\cup e} H^3(N) = \mathbb{Z}.$$

Here $p_* : H^2(M) \rightarrow H^1(N)$ is the map given by integration along the fiber. The same sequence can be considered for cohomology with real coefficients.

The first group of results, in the spirit of [FV06b], is aimed at characterizing the topology of symplectic 4-manifolds with a free circle action. The starting point is the following result, that appears (with different generality) in [Th76], [Bou88] and [FGM91].

Proposition 1. *Let $p : M \rightarrow N$ be a principal S^1 -bundle with Euler class e . Let $\phi \in H^1(N)$ be a fibered class such that $\phi \cup e = 0$. Then M can be endowed with a symplectic form ω with the property that $p_*[\omega] = \phi$.*

A folklore conjecture posits that the converse of the corollary holds (cf. e.g. [Kr99] and [Bal01]). Under subgroup separability assumptions, we can prove that this is indeed the case.

Theorem 2. *Let (M, ω) be a symplectic 4-manifold admitting a free circle action. Assume that $\phi = p_*[\omega] \in H^1(N)$ is a primitive class. Furthermore assume that the class dual to ϕ can be represented by a connected incompressible embedded surface Σ such that $\pi_1(\Sigma)$ is separable in $\pi_1(N)$. Then (N, ϕ) fibers over S^1 .*

(This statement depends, in part, on the geometrization conjecture, that is required to show that the orbit space N is prime.)

Concerning this statement, note that for any symplectic structure ω on M we have $p_*([\omega]) \neq 0$ (cf. Section 2), hence, using openness of the symplectic condition and scaling suitably, the integrality and primitiveness conditions on $p_*[\omega]$ are not restrictive. Also, recall that a finitely generated subgroup $A \subset \pi_1(N)$ is said to be *separable* if, for all $g \in \pi_1(N) \setminus A$, there exists an epimorphism onto a finite group $\alpha : \pi_1(N) \rightarrow G$ such that $\alpha(g) \notin \alpha(A)$. In particular, the separability assumption of the theorem is satisfied if N has vanishing Thurston norm ([LN91]) and, conjecturally ([Th82]), for all hyperbolic manifolds. (In the special case that N has vanishing Thurston norm this result has independently been obtained also by Bowden [Bow07], building on ideas of [FV06b].) With similar methods, we prove that the statement holds true unconditionally in the case where N is a graph manifold (cf. Proposition 3.7 and Theorem 5.1).

As mentioned above, the present paper covers the case where the Euler class $e \in H^2(N)$ of the S^1 -fibration $p : M \rightarrow N$ is non-trivial; the product case is covered in [FV06b].

Next, we address the related question of which manifolds as above can be endowed with a symplectic structure with trivial canonical class. Our result is the following.

Theorem 3. *Let M be a manifold with free circle action. Then M admits a symplectic structure with trivial canonical class if and only if it is a T^2 -bundle over T^2 .*

In fact, we will prove that if M is symplectic, the orbit space N is a torus bundle over S^1 , which implies, with further considerations, the stated result. The ‘if’ part of Theorem 3 follows then easily from Proposition 1. The proof of Theorem 3 follows by combining

separability of torus subgroups with a rather unexpected application of the Lubotzky alternative for finitely generated linear groups. Our result, which groups together Theorem 4.1 and Corollary 5.2, confirms that all symplectic manifolds with trivial canonical class admitting a free circle action are contained in Table 1 of [Li06a].

The previous results ascertain what is (likely) the topology of 4-manifolds with free circle action that admit a symplectic structure. In particular we obtain results on the image of the symplectic cone under the map $p_* : H^2(M, \mathbb{R}) \rightarrow H^1(N, \mathbb{R})$. The second group of results is devoted to the study of the symplectic cone for these manifolds, and the proof that the cohomology classes in that cone can be realized by S^1 -invariant symplectic forms.

The main result is the following existence theorem, that generalizes the aforementioned constructions of [Th76], [Bou88] and [FGM91].

Theorem 4. *Let M be a 4-manifold admitting a free circle action. Let $\psi \in H^2(M; \mathbb{R})$ such that $\psi^2 > 0 \in H^4(M; \mathbb{R})$ and such that $p_*(\psi) \in H^1(N; \mathbb{R})$ can be represented by a non-degenerate closed 1-form. Then there exists an S^1 -invariant symplectic form ω on M with $[\omega] = \psi \in H^2(M; \mathbb{R})$.*

Proposition 1 clearly follows from this theorem. It is also worth mentioning that S^1 -invariant symplectic forms on M have first been constructed by Bouyakoub [Bou88]. More precisely, in [Bou88] it was shown that given ψ as in the theorem, there exists an S^1 -invariant symplectic form ω with $p_*([\omega]) = p_*(\psi)$. In the case that $p_*(\psi)$ is rational our theorem can also be deduced from [FGM91].

Combining the above results with well-known properties of the Thurston norm ball we get the following result.

Theorem 5. *Let M be a 4-manifold with free S^1 -action such that the orbit space N is a graph manifold or has vanishing Thurston norm. Then a class $\psi \in H^2(M; \mathbb{R})$ can be represented by a symplectic form if and only if $\psi^2 > 0$ and $p_*(\psi) \in H^1(N; \mathbb{R})$ lies in the open cone on a fibered face of the Thurston norm ball. Furthermore, we can represent such ψ by an S^1 -invariant symplectic.*

(Recall that by [Th86] the class $\phi \in H^1(N; \mathbb{R})$ lies in the open cone on a fibered face if and only if ϕ can be represented by a non-singular 1-form. Also, when the Thurston norm vanish, every nonzero element of $H^1(N; \mathbb{R})$ is considered to lie in the open cone on a fibered face.) In particular this determines completely the symplectic cone for such a class of manifolds, recovering (for the case of vanishing Thurston norm) results of Geiges (see [Ge92]). Note that we are not claiming that *any* symplectic form is isotopic, or even homotopic to an S^1 -invariant form, although this might be the case.

This paper is structured as follows. In the first part, Sections 2 to 4 cover the case where the Euler class is nontorsion: Section 2 is devoted to summarizing some standard results about the topology of M that will be useful in what follows; Section 3 discusses Seiberg-Witten theory in this context, and contains the proof of the Theorem 2 in that case, using the constraints on (twisted) Alexander polynomials arising from Seiberg-Witten theory on symplectic manifolds; Section 4 is dedicated to the case of trivial canonical class. Section 5 covers the case where the Euler class is torsion. Section 6 contains some explicit examples to show the implications of our results. In the second part, corresponding to Section 7, we discuss the construction of symplectic structures on manifolds M for which the orbit space

fibers over the circle, leading to Theorems 4 and 5.

Convention. All maps are assumed to be C^∞ unless stated otherwise. All manifolds are assumed to be connected, compact, closed and orientable. All homology and cohomology groups are with integral coefficients, unless it says specifically otherwise.

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2. ALGEBRAIC TOPOLOGY OF 4-MANIFOLDS WITH A FREE CIRCLE ACTION

The purpose of this section is to recall some elementary facts about 4-manifolds admitting a free circle action with nontorsion Euler class, as they will frequently be used in what follows. The existence of a free circle action on M implies that M is a principal S^1 -bundle, so that there is a projection map $p : M \rightarrow N$ where we denote by N the orbit space of the free circle action. This principal bundle is determined by its Euler class $e \in H^2(N)$, and for either integer or real coefficients we have the Gysin sequence

$$(2) \quad \cdots \rightarrow H^{q-2}(N) \xrightarrow{\cup e} H^q(N) \xrightarrow{p^*} H^q(M) \xrightarrow{p_*} H^{q-1}(N) \rightarrow \cdots$$

where $p_* : H^q(M) \rightarrow H^{q-1}(N)$ denotes integration along the fiber. Assuming that e is not a torsion class it is easy to verify, using this sequence, that $b_1(M) = b_1(N)$ and $b_2(M) = 2b_1(N) - 2$. We want to determine the intersection form on $H^2(M)/\text{Tor}$. We will denote, for either M or N , the pairing $\langle \alpha \cup \beta, [\cdot] \rangle$ by $\alpha \cdot \beta$. It is convenient to break down the Gysin sequence as

$$(3) \quad 0 \rightarrow \langle e \rangle \rightarrow H^2(N) \xrightarrow{p^*} H^2(M) \xrightarrow{p_*} \ker(\cdot e) \rightarrow 0,$$

where we have denoted by $\langle e \rangle$ the cyclic subgroup of $H^2(N)$ generated by the Euler class and by $\ker(\cdot e)$ the subgroup of $H^1(N)$ whose pairing with the Euler class vanishes. Choose a basis $\{\phi_i, i = 1, \dots, b_1(N) - 1\}$ for $\ker(\cdot e)$ and denote by $\{\psi_j, j = 1, \dots, b_1(N) - 1\}$ a set of elements of $H^2(N)$ with the property that $\phi_i \cdot \psi_j = \delta_{ij}$: the existence of such a set is granted by the fact that we can identify, using Poincaré duality, $H^1(N)$ with $\text{Hom}(H^2(N); \mathbb{Z})$. By slight abuse of notation, denote by the same symbol the image of these elements in $H^2(N)/\text{Tor}$, and denote by Φ_i a representative of the inverse image of ϕ_i in $H^2(M)$. Note that

$$\{p^*\psi_j, \Phi_i, j = 1, \dots, b_1(N) - 1, i = 1, \dots, b_1(N) - 1\}$$

is a basis for $H^2(M)/\text{Tor}$. Since $p^*\psi_j \cdot p^*\psi_k = 0$, $\Phi_i \cdot p^*\psi_j = \phi_i \cdot \psi_j = \delta_{ij}$ we see that the intersection form on $H^2(M)/\text{Tor}$ has the form

$$(4) \quad \begin{pmatrix} 0 & I \\ I & A \end{pmatrix}$$

where the submatrices have rank $b_1(N) - 1$ and A is some symmetric matrix with entries $\Phi_i \cdot \Phi_j$. At this point it is an easy exercise in linear algebra to see that $b_2^\pm(M) = b_1(N) - 1$ and $\sigma(M) = 0$. Finally, the long exact homotopy sequence of the fibration gives

$$(5) \quad 0 \rightarrow \pi_2(M) \rightarrow \pi_2(N) \rightarrow \mathbb{Z} \rightarrow \pi_1(M) \rightarrow \pi_1(N) \rightarrow 1,$$

while $\pi_i(M) = \pi_i(N)$ for $i > 2$.

Let $\alpha : \pi_1(N) \rightarrow G$ be a homomorphism to a finite group. We denote by $\pi : N_G \rightarrow N$ the regular G -cover of N . It is well known that $\pi : H_1(N_G; \mathbb{Q}) \rightarrow H_1(N; \mathbb{Q})$ is surjective, in particular $b_1(N_G) \geq b_1(N)$. If $\pi : M \rightarrow N$ is a circle bundle with Euler class $e \in H^2(N)$ then α determines a regular G -cover of M that we will denote (with slight abuse of notation) $\pi : M_G \rightarrow M$. These covers are related by the commutative diagram

$$(6) \quad \begin{array}{ccc} M_G & \xrightarrow{\pi} & M \\ \downarrow & & \downarrow \\ N_G & \xrightarrow{\pi} & N \end{array}$$

where the principal S^1 -fibration $p_G : M_G \rightarrow N_G$ has nontorsion Euler class $e_G = \pi^*e \in H^2(N_G)$. Observe that it follows immediately from the fact that $\pi : H_1(N_G; \mathbb{Q}) \rightarrow H_1(N; \mathbb{Q})$ is surjective that e is non-torsion if and only if e_G is non-torsion.

3. CONSTRAINTS FROM SEIBERG-WITTEN THEORY

3.1. Seiberg-Witten theory for symplectic manifolds with a circle action. In this section we will use Seiberg-Witten theory to prove that, under suitable assumptions of subgroup separability, the orbit space N of a symplectic manifold M admitting a free circle action with nontorsion Euler class is fibered. We will start with the following proposition, that asserts that N is Haken, in particular a $K(\pi, 1)$ space and, because of (5), so is M .

Proposition 3.1. *Let M be a symplectic 4-manifold admitting a free circle action with nontorsion Euler class $e \in H^2(N)$, where N is the orbit space. Then N is a Haken manifold with $b_1(N) > 1$.*

Proof. The proof of primeness of N , that generalizes the product case discussed in [McC01]), is based on Taubes' results on Seiberg-Witten invariants of symplectic manifolds, and exploits the residual finiteness of the fundamental group of 3-manifolds; a detailed argument appears in [Bow07]. The condition that M is symplectic entails that $b_2^+(M) > 0$; by the discussion in the previous section, it follows that $b_1(N) > 1$, hence that N is Haken. \square

To further use Seiberg-Witten theory, we need more information on the Seiberg-Witten invariants of M . The essential ingredient for this is the fact that the Seiberg-Witten invariants of M are related to the Alexander polynomial of N . Baldridge proved the following result, that combines Corollaries 25 and 27 of [Bal03] (cf. also [Bal01]), to which we refer the reader for definitions and results for Seiberg-Witten theory in this set-up:

Theorem 3.2. (Baldridge) *Let M be a 4-manifold admitting a free circle action with nontorsion Euler class $e \in H^2(N)$, where N is the orbit space. Then the Seiberg-Witten invariant $SW_M(\kappa)$ of a class $\kappa = p^*\xi \in p^*H^2(N) \subset H^2(M)$ is given by the formula*

$$(7) \quad SW_M(\kappa) = \sum_{\xi \in (p^*)^{-1}(\kappa)} SW_N(\xi) = \sum_{\xi' - \xi \equiv 0(e)} SW_N(\xi') \in \mathbb{Z},$$

in particular when $b_2^+(M) = 1$ it is independent on the chamber in which it was calculated. Moreover, if $b_2^+(M) > 1$, these are the only basic classes.

(For completeness, we remark that if $b_2^+(M) = 1$, the Seiberg-Witten invariants of a class of $H^2(M)$ that is not a pull-back can be nonzero in one of the two chambers, and is determined by the wall-crossing contribution.)

In the formula above, $SW_N(\xi)$ is the 3-dimensional SW-invariant of a class $\xi \in H^2(N)$, and the effect of the twisting of the S^1 -fibration, measured by the class $e \in H^2(N)$, is to wrap up the contribution of all 3-dimensional basic classes of N that have the same image in $H^2(M)$, i.e. that differ by a multiple of e . As usual, we can package the above invariants in terms of a Seiberg-Witten polynomial.

We will use Taubes' constraints on the Seiberg-Witten invariants for symplectic manifolds and Baldridge's formula to get a constraint on the twisted Alexander polynomials of the orbit space N of a symplectic manifold M admitting a circle action as above. Before doing so, we need some definitions and results. Recall from [FV06a, Section 3.1] that, given an epimorphism to a finite group $\alpha : \pi_1(N) \rightarrow G$, the 1-variable twisted Alexander $\Delta_{N,\phi}^\alpha \in \mathbb{Z}[t^{\pm 1}]$ associated to a class $\phi \in H^1(N)$ is the order of the first twisted Alexander module $H_1(N; \mathbb{Z}[G][t^{\pm 1}])$. This polynomial is defined up to a unit of $\mathbb{Z}[t^{\pm 1}]$; this indeterminacy will be irrelevant (and therefore omitted) in what follows. If α is the homomorphism to the trivial group then we denote the resulting polynomial by $\Delta_{N,\phi}$. We refer to [FV06a, Section 3.1] for more details.

Let M be a manifold admitting a free circle action with orbit space N and nontorsion Euler class $e \in H^2(N)$ and assume that M can be endowed with a symplectic form $\omega \in \Omega^2(M)$. As $[\omega]^2 > 0$ it follows from Section 2 that $[\omega] \notin p^*(H^2(N, \mathbb{R}))$, and in particular $p_*[\omega] \neq 0 \in H^1(N; \mathbb{R})$. Using openness of the symplectic condition, we can assume that $[\omega] \in H^2(M; \mathbb{R})$ lies in the rational lattice (identified with) $H^2(M; \mathbb{Q})$. After suitably scaling ω by a rational number if needed, the class $p_*[\omega]$ is then (the image of) a primitive (in particular, nonzero) class in $H^1(N)$ that we denote by ϕ .

We will also need the following result regarding the canonical class of the symplectic structure.

Proposition 3.3. *Let (M, ω) be a symplectic manifold admitting a free circle action with nontorsion Euler class $e \in H^2(N)$, where N is the orbit space. Then the canonical class $K \in H^2(M)$ of the symplectic structure is the pull-back of a class $\zeta \in H^2(N)$, well-defined up to the addition of a multiple of e .*

Proof. If $b_2^+(M) > 1$ this is a straightforward consequence of Theorem 3.2, as the canonical class by [Ta94] is a basic class of M , hence must be the pull-back of a class of $H^2(N)$. The case of $b_2^+(M) = 1$ can be similarly obtained with a careful analysis of the chamber structure of the Seiberg-Witten invariants for classes that are not pull-back, but it is possible to use a quicker argument. First, observe that, starting from a closed curve in N representing a suitable element of $H_1(N)$, we can identify a torus $T \subset M$ of self-intersection zero, representing the generator of a cyclic subgroup in the image of the map $H_1(N) \rightarrow H_2(M)$ in the homology Gysin sequence, that satisfies $\omega \cdot [T] > 0$. Second we can assume, by [Liu96], that $K \cdot \omega \geq 0$. Otherwise, M would be a rational or ruled surface, which cannot happen as these satisfy $\pi_2 \neq 0$. As both signature and Euler characteristic of M vanish, $K^2 = 2\chi(M) + 3\sigma(M) = 0$. Omitting the case of K torsion, where the statement is immediate, we deduce that both K and (the Poincaré dual of) $[T]$ lie in the closure of the forward positive cone in $H^2(M, \mathbb{R})$ determined by ω . The light-cone Lemma (see e.g.

[Liu96]) asserts at this point that $K \cdot [T] \geq 0$. On the other hand, the adjunction inequality of Li and Liu (see [LL95]) gives $K \cdot [T] \leq 0$, hence $K \cdot [T] = 0$. It now follows that K is a multiple of $PD([T])$, in particular the pull-back of a class on N . \square

We are in position now to use Equation (7) to obtain the following. (Note that we use Proposition 3.3 to formulate the statement.)

Theorem 3.4. *Let (M, ω) be a symplectic manifold admitting a free circle action with nontorsion Euler class and let $\phi = p_*[\omega] \in H^1(N)$ a primitive class on the orbit space N . Then for all epimorphisms $\alpha : \pi_1(N) \rightarrow G$ to a finite group the twisted Alexander polynomial $\Delta_{N, \phi}^\alpha \in \mathbb{Z}[t^{\pm 1}]$ is monic of Laurent degree*

$$\deg \Delta_{N, \phi}^\alpha = |G| \zeta \cdot \phi + 2 \operatorname{div} \phi_G.$$

Here, $\zeta \in H^2(N)$ is a class whose pull-back to M gives the canonical class of M . Furthermore ϕ_G denotes the restriction of $\phi : \pi_1(N) \rightarrow \mathbb{Z}$ to $\operatorname{Ker} \alpha$ and $\operatorname{div} \phi_G$ stands for the divisibility of ϕ_G .

Proof. Our goal is to apply Taubes' results ([Ta94, Ta95]) on the Seiberg–Witten invariants of M to impose constraints on the twisted Alexander polynomials of N . We will first analyze the constraints on the ordinary 1-variable Alexander polynomial $\Delta_{N, \phi}$. By [FV06a, Proposition 3.6] we can write this polynomial as

$$(8) \quad \Delta_{N, \phi} = (t^{\operatorname{div} \phi} - 1)^2 \cdot \sum_{g \in H} a_g t^{\phi(g)} \in \mathbb{Z}[t^{\pm 1}],$$

where H is the maximal free abelian quotient of $\pi_1(N)$ and $\Delta_N = \sum_{g \in H} a_g \cdot g \in \mathbb{Z}[H]$ is the ordinary multivariable Alexander polynomial of N . By Meng and Taubes ([MT96]) the latter is related to the Seiberg–Witten invariants of N via the formula

$$(9) \quad \sum_{g \in H} a_g \cdot g = \pm \sum_{\xi \in H^2(N)} SW_N(\xi) \cdot \frac{1}{2} f(\xi) \in \mathbb{Z}[H],$$

where f denotes the composition of Poincaré duality with the quotient map $f : H^2(N) \cong H_1(N) \rightarrow H$ and, as $f(\xi)$ has even divisibility for all 3-dimensional basic classes $\xi \in \operatorname{supp} SW_N$, multiplication by $\frac{1}{2}$ is well-defined. Using this formula, we can write

$$(10) \quad \Delta_{N, \phi} = \pm (t^{\operatorname{div} \phi} - 1)^2 \sum_{\xi \in H^2(N)} SW_N(\xi) t^{\frac{1}{2} \phi \cdot \xi}.$$

We will use now Equation (7) to write $\Delta_{N, \phi}$ in terms of the 4-dimensional Seiberg–Witten invariants of M . In order to do so, observe that for all classes $\xi \in H^2(N)$ we can write $\xi \cdot \phi = \xi \cdot p_* \omega = p^* \xi \cdot \omega = \kappa \cdot \omega$ where $\kappa = p^* \xi$. Grouping together the contributions of the 3-dimensional basic classes in terms of their image in $H^2(M)$, and using (7) we get

$$\begin{aligned} \Delta_{N, \phi} &= \pm (t^{\operatorname{div} \phi} - 1)^2 \sum_{\kappa \in p^* H^2(N)} \sum_{\xi \in (p^*)^{-1}(\kappa)} SW_N(\xi) t^{\frac{1}{2} \phi \cdot \xi} \\ &= \pm (t^{\operatorname{div} \phi} - 1)^2 \sum_{\kappa \in p^* H^2(N)} SW_M(\kappa) t^{\frac{1}{2} \kappa \cdot \omega}. \end{aligned}$$

Taubes' constraints, applied to the symplectic manifold (M, ω) , assert that if $K \in H^2(M)$ is the canonical class, then $SW_M(-K) = 1$. Moreover, among all basic classes $\kappa \in H^2(M)$, we have

$$(11) \quad -K \cdot \omega \leq \kappa \cdot \omega,$$

with equality possible only for $\kappa = -K$. (When $b_2^+(M) = 1$, this statement applies to the Seiberg-Witten invariants evaluated in Taubes' chamber, but as remarked in Theorem 3.2 this specification is not a concern in our situation.) It now follows immediately from Proposition 3.3 that $\Delta_{N, \phi}$ is a monic polynomial, and remembering the symmetry of SW_N (or $\Delta_{N, \phi}$), we see that its Laurent degree is $d = K \cdot \omega + 2\text{div}\phi = \zeta \cdot \phi + 2\text{div}\phi$.

Consider now any symplectic 4-manifold M satisfying the hypothesis of the statement. Take an epimorphism $\alpha : \pi_1(N) \rightarrow G$ and denote, as in Section 2, by N_G and M_G the associated G -covers of N and M respectively. We will bootstrap the constraint on the ordinary Alexander polynomials to the twisted Alexander polynomials associated to $\alpha : \pi_1(N) \rightarrow G$. As (M, ω) is symplectic, M_G inherits a symplectic form $\omega_G := \pi^*\omega$ with canonical class $K_G := \pi^*K$, that is easily shown to satisfy the condition $\phi_G := (p_G)_*[\omega_G] = \pi^*\phi \in H^1(N_G)$. We can therefore apply the results of the previous paragraph to the pair (N_G, ϕ_G) to get a constraint for Δ_{N_G, ϕ_G} . This, together with the relation $\Delta_{N, \phi}^\alpha = \Delta_{N_G, \phi_G}$ proven in [FV06a, Lemma 3.3] and some straightforward calculations show that $\Delta_{N, \phi}^\alpha$ is monic of the degree stated. \square

Remark. In [FV06a, Proposition 4.4] we obtained a similar result for $M = S^1 \times N^3$. However, in that case, using results based on the refined adjunction inequality for $S^1 \times N$ proved by Kronheimer in [Kr99], we get the additional constraint that $\zeta \cdot \phi = \|\phi\|_T$, i.e. altogether that $\Delta_{N, \phi}^\alpha$ determines the Thurston norm of ϕ . If (N, ϕ) is a fibered class, we can assume $\zeta \in H^2(N)$ to be (up to sign) the Euler class of the fibration, and this information is encoded in $\Delta_{N, \phi}^\alpha$. It is not clear whether Kronheimer's results hold in the case of general circle bundles over 3-manifolds (but cf. [FV07]).

3.2. Proof of Theorem 2 for non-torsion Euler class. Our goal is to show that Theorem 3.4 implies that the pair (N, ϕ) fibers. As stated in the introduction, we will be able to do so under the same assumptions of subgroup separability that we used in [FV06b]. More precisely we will assume the separability in $\pi_1(N)$ of the image of the fundamental group $\pi_1(\Sigma)$ of an incompressible representative Σ of the class Poincaré dual to ϕ .

Theorem 3.5. *Let (M, ω) be a symplectic manifold admitting a free circle action with nontorsion Euler class. If $\phi = p_*[\omega] \in H^1(N)$ is a primitive class on the orbit space N and if the class dual to ϕ can be represented by a connected incompressible embedded surface Σ such that $\pi_1(\Sigma)$ is separable in $\pi_1(N)$, then (N, ϕ) fibers over S^1 .*

Proof. Assume, by contradiction, that (N, ϕ) is not fibered. Our goal is to apply Theorem 3.4 to show that this is inconsistent, assuming separability, with the hypothesis that M is symplectic.

Consider the exterior $N \setminus \nu\Sigma$. This is a connected manifold with boundary components two copies Σ_\pm of Σ . We choose either copy, that we will denote simply by Σ . By incompressibility of Σ the inclusion induced maps $\pi_1(\Sigma) \rightarrow \pi_1(N \setminus \Sigma) \rightarrow \pi_1(N)$ are injections,

we can therefore view $\pi_1(\Sigma)$ as a subgroup of $\pi_1(N \setminus \nu\Sigma)$ and $\pi_1(N \setminus \nu\Sigma)$ as a subgroup of $\pi_1(N)$.

Since N is irreducible by Proposition 3.1 and as ϕ is not fibered, Stallings' Theorem (cf. [St62] and [He76]) asserts that $\pi_1(\Sigma)$ is in fact a proper subgroup of $\pi_1(N \setminus \Sigma)$. By assumption we can find a homomorphism $\alpha : \pi_1(N) \rightarrow G$ to a finite group with $\alpha(\pi_1(\Sigma)) \subsetneq \alpha(\pi_1(N \setminus \nu\Sigma))$. But it follows from Theorem 4.2 of [FV06b] that the twisted Alexander polynomial $\Delta_{N,\phi}^\alpha$ vanishes, which contradicts Theorem 3.4. \square

In [FV06b] we have discussed various cases where the separability conditions of Theorem 3.5 are known to hold true, and we refer to there the interested reader. Two cases, however, deserve mentioning. The first is that it has long been conjectured that for hyperbolic manifolds separability holds with respect to all finitely generated subgroups (in that case we say that $\pi_1(N)$ is *LERF*). The second case, in a sense the diametrical opposite, covers manifolds with vanishing Thurston norm. We have the following result (cf. also [Bow07]).

Corollary 3.6. *Let (M, ω) be a symplectic manifold admitting a free circle action with nontorsion Euler class and assume that the orbit space N has vanishing Thurston norm. Then (N, ϕ) fibers over S^1 for all non-trivial classes $\phi \in H^1(N)$.*

Proof. As N is Haken and the Thurston norm of N vanishes, every primitive class can be represented by an incompressible torus. By [LN91], the fundamental group of an incompressible torus is separable in a Haken manifold. Hence it follows from Theorem 3.5 that N admits a fibration in tori; but then it is well-known that all nonzero classes are fibered. \square

Note for future reference that, for all manifolds of Corollary 3.6, the canonical class is trivial.

When N is a graph manifold, it is known that, at least in general, $\pi_1(N)$ is not LERF (while it is still unknown whether it satisfies the surface subgroup separability conditions of Theorem 3.5). In spite of this potential setback, in [FV06b, Corollary 5.6] we showed, using abelian subgroup separability and the classification of incompressible surfaces in Seifert fibered spaces, that if N is a graph manifold, then (N, ϕ) fibers if and only if $\Delta_{N,\phi}^\alpha$ is non-zero for any epimorphisms $\alpha : \pi_1(N) \rightarrow G$ to a finite group. As a consequence of this and Theorem 3.4 we deduce the following proposition.

Proposition 3.7. *Let (M, ω) be a symplectic manifold admitting a free circle action with nontorsion Euler class and assume that the orbit space N is a graph manifold. Let $\phi = p_*[\omega] \in H^1(N)$ be a primitive class, then (N, ϕ) fibers over S^1 .*

4. THE CASE OF TRIVIAL CANONICAL CLASS

Although it is pleasant that Theorem 3.5 reduces the problem of determining which manifolds with a free circle action admit a symplectic structure to classical conjectures in 3-dimensional topology, it is appropriate to keep a critical approach to this statement and try to extend its range of applications to results that hold unconditionally.

In this section we will address the question of which manifolds admitting a free circle action with nontorsion Euler class can be endowed with a symplectic structure with trivial canonical class, a problem that, as observed, is strictly related with Proposition 3.6. The class of symplectic manifolds with $K = 0$ has been studied by various authors, and the list

of the known examples is detailed in [Li06a]. These examples are either the $K3$ surface or T^2 -bundles over T^2 (which are always symplectic, with trivial canonical). Our goal is to confirm that, when it comes to manifolds admitting a free circle action, the list is complete. In the case of symplectic manifolds of the form $S^1 \times N$ this question was completely solved, using results from [Vi03], in [FV06b]. One of the main ingredients of that result is the idea, due to Kronheimer, that for manifolds of the form $S^1 \times N$ the refined adjunction inequality established in [Kr99] allows one to constrain the Thurston norm of a 3-manifold in terms of the canonical class. When $K = 0$ this constraint translates into the fact that N must have vanishing Thurston norm, and at this point a result similar to Corollary 3.6 completes the argument. In the case we are studying, instead, it is not known whether a refined adjunction inequality similar to that established in [Kr99] holds, and we are forced to gather information by other means. We will succeed in doing so by extending the approach of Section 2 of [FV06b], using as new topological ingredient a consequence of the Lubotzky alternative for finitely generated linear groups.

We remark that, if we assume that Theorem 3.5 holds unconditionally, N must in fact have vanishing Thurston norm: the condition $\deg \Delta_{N,\phi} = 2\operatorname{div} \phi$ (that arises when we specialize Proposition 3.4 to the case $K = 0$) is possible, for a fibered ϕ , only if ϕ is represented by a torus, as for a fibered class the Thurston and the Alexander norm must coincide. Even without that assumption, we have the following result.

Theorem 4.1. *Let M be a manifold endowed with a free circle action with nontorsion Euler class. Then M admits a symplectic structure with trivial canonical class if and only if it is a T^2 -bundle over T^2 .*

Proof. We will start by showing that the condition $K = 0$ implies that the virtual Betti number $vb_1(N)$ of the orbit space N (i.e. the *sup* of the Betti number of all covers of N) satisfies the condition $vb_1(N) \leq 3$. Taubes' constraints imply that $K = 0$ satisfies $SW_M(0) = 1$ and, using the symmetry of the Seiberg-Witten invariants of N and Equation (7), we see that this is the only basic class (when $b_2^+(M) = 1$, this is true in $p^*H^2(N)$, applying the usual *caveat* for this case). We can then compute the sum of the coefficients of the Seiberg-Witten invariant of N as

$$(12) \quad \sum_{\xi \in H^2(N)} SW_N(\xi) = \sum_{\kappa \in p^*H^2(N)} SW_M(\kappa) = 1.$$

Now, for all 3-manifolds with $b_1(N) > 1$, the sum of the coefficients of the Seiberg-Witten polynomial equals by (9) the sum of the coefficients of the Alexander polynomial, and the latter vanishes when $b_1(N) > 3$ (see [Tu02, Section II.5.2 and Theorem IX.2.2]). Equation (12) requires therefore that $b_1(N) \leq 3$. Repeating this argument for all covers of N (for which the Euler class is necessarily nontorsion and the canonical zero) gives the desired bound on $vb_1(N)$.

We want to show that this condition entails that either N is a torus bundle, or N is hyperbolic. In fact, if N is a torus bundle, all its finite covers are torus bundles, so $vb_1(N) \leq 3$. Otherwise, if N has a non-trivial JSJ decomposition, or is Seifert fibered, it contains an incompressible torus T that is not a fiber. (This is obvious if N has a non-trivial JSJ decomposition, but it is true also when N is Seifert fibered, as Seifert fibered manifolds without incompressible tori must have $b_1 \leq 1$ (cf. [Ja80, p. 89ff]).) The fundamental

group of T is separable in $\pi_1(N)$, so we can proceed as follows. If T separates N in two components, up to passing to a suitable cover, it lifts to a nonseparating torus, so we can restrict ourselves to the latter case. If the torus T is nonseparating, it cannot be a fiber, as otherwise all nonzero cohomology classes would be fibered, and so necessarily would be N . Hence, by [Ko87], $vb_1(N) = \infty$.

To complete the proof that N is fibered, it remains to show that a case where N is a hyperbolic manifold with $vb_1(N) \leq 3$ can be excluded. It is widely expected (and verified for the arithmetic case, see [CLR06]) that hyperbolic manifolds with positive Betti number have $vb_1(N) = \infty$, so it is quite possible that that case is taken care of by the previous result, but even if we lack a general proof of this fact, we will be able to explicitly rule out that case too.

We start by observing that, as consequence of [Tu02, Section II.5.2 and Theorem IX.2.4], if the homology group $H_1(N, \mathbb{F}_p)$ has rank $b_1(N, \mathbb{F}_p) > 3$, then the sum of the coefficients of the Alexander polynomial vanishes *mod* p . Now a consequence of the Lubotzky alternative (cf. [LS03, Corollary 16.4.18]) asserts that if $\pi_1(N)$ is a finitely generated linear group, then either $\pi_1(N)$ is virtually soluble or, for any prime p , $vb_1(N, \mathbb{F}_p) = \infty$, i.e. N admits finite covers with arbitrarily large first homology with coefficients in \mathbb{F}_p (see also [La05, Theorem 1.3]). Since N is hyperbolic, its fundamental group is of course linear. By [EM72, Theorem 4.5] the only (orientable) 3-manifolds with positive Betti number and soluble fundamental group are torus bundles, whence the first condition cannot occur. It follows that for any prime p , there exists a cover of N (even regular, by [La05, Theorem 5.1]) whose Alexander polynomial has sum of coefficients that vanishes *mod* p . This entails that the corresponding cover of M violates Equation (12), hence the possibility of a hyperbolic N is excluded.

To complete the proof, observe that a torus bundle with $b_1(N) \geq 2$ is also an S^1 -bundle over T^2 (see e.g. [Hat]), so that M is a T^2 -bundle over T^2 . These manifolds admit a symplectic structure by [FGM91] (cf. also [Ge92]), and it is not too difficult to verify that $K = 0$. \square

- Remark.*
- (1) Note that the constraint on the virtual Betti number established in Theorem 4.1 proves Conjecture 1.1 of [Li06a] for the class of manifolds under question. (This conjecture has been confirmed in full generality in [Bau06] and [Li06b], using Bauer-Furuta's refinement of Seiberg-Witten invariants.)
 - (2) The proof of Theorem 4.1 applies, *mutatis mutandis*, to the product case, making it unnecessary there as here to use the refined adjunction inequality or Donaldson's theorem on the existence of symplectic representatives of (sufficiently high multiples of) the dual of $[\omega]$, that are used instead in [Kr99] and [Vi03].
 - (3) Note that the statement of Theorem 4.1 covers in fact all symplectic manifolds with *torsion* canonical class (a class that *a priori* could be broader, when $b_2^+(M) = 1$, than the case of trivial canonical class). In fact, if $b_2^+(M) = 1$, $b_1(M) = 2$, and K is torsion, McDuff and Salamon show in [MS96] that K is actually trivial. (This can actually be verified, in the case at hand, using Taubes' constraints and the symmetry of SW_N .) So Theorem 4.1 covers all symplectic manifolds with Kodaira dimension 0, in the notation of [Li06a] (M is symplectically minimal, as it is aspherical).

5. THE TORSION CASE

In this section we will treat the case of a symplectic 4-manifold admitting a circle action with orbit space N such that the Euler class is torsion. We will adopt the notation of the previous sections, expecting that the minor differences for this case will provide no inconveniences to the reader.

Observe that, exactly as in Proposition 3.1, the orbit space N is either irreducible or $N = S^1 \times S^2$ in correspondence of which we have the product symplectic manifold $T^2 \times S^2$; in this case N is obviously fibered, and we will omit it in what follows.

At this point, we could proceed as in Theorem 3.4, using the results of [Bal03] for the relation between the Seiberg-Witten invariants of M and N in case of torsion Euler class. This does not present particular conceptual difficulties but requires, in the case of $b_2^+(M) = b_1(N) = 1$, a detailed bookkeeping of the chamber dependence of the Seiberg-Witten invariants for both M and N , that would impose on us a somewhat long detour. Instead of following that path, it is simpler to use a straightforward algebro-topological observation to reduce the problem to the product case treated in [FV06b]. We have the following

Theorem 5.1. *Let (M, ω) be a symplectic manifold admitting a free circle action with torsion Euler class. Let $\phi = p_*[\omega] \in H^1(N)$ be a primitive class on the orbit space $N \neq S^1 \times S^2$. Assume that one of the following holds:*

- (1) *the class dual to ϕ can be represented by a connected incompressible embedded surface Σ such that $\pi_1(\Sigma)$ is separable in $\pi_1(N)$, or*
- (2) *N is a graph manifold, or*
- (3) *the canonical class K is trivial,*

then (N, ϕ) fibers over S^1 .

Proof. Assume, by contradiction, that (N, ϕ) is not fibered. If the Euler class $e \in H^2(N)$ is trivial, M is a product $S^1 \times N$, and this case is discussed in [FV06b]. So we assume that $e \neq 0$, so that in particular $\text{Tor}(H_1(N))$ is non-trivial.

We first consider case (1). It follows by assumption that there exists an epimorphism $\alpha : \pi_1(N) \rightarrow G$ onto a finite group with the property that $\alpha(\pi_1(\Sigma)) \subsetneq \alpha(\pi_1(N \setminus \nu\Sigma))$ (cf. also the proof of Theorem 3.5). This entails by [FV06b, Theorem 4.2] that $\Delta_{N, \phi}^\alpha = 0$, but we cannot use directly this information without establishing an explicit relation between twisted Alexander polynomials of N and Seiberg-Witten invariants on M that, as observed above, requires a little work when $b_1(N) = 1$. This is however only a minor setback, as we can proceed as follows. Denote by T an abelian group isomorphic to $\text{Tor}(H_1(N))$, and pick a map $H_1(N) \rightarrow T$ such that the induced map

$$\text{Tor}(H_1(N)) \rightarrow H_1(N) \rightarrow T$$

is an isomorphism. Denote by $\beta : \pi_1(N) \rightarrow H_1(N) \rightarrow T$ the corresponding map from the fundamental group of N . Out of that, we can define a map

$$\begin{aligned} \gamma : \pi_1(N) &\longrightarrow G \times T \\ g &\longmapsto (\alpha(g), \beta(g)). \end{aligned}$$

Let $\Gamma := \gamma(\pi_1(N)) \subset G \times T$, and note that there is a well-defined epimorphism $\Gamma \rightarrow T$. We claim that, denoting as usual by $\pi : N_\Gamma \rightarrow N$ the associated regular cover, and by

$\pi : M_\Gamma \rightarrow M$ its 4-dimensional counterpart, M_Γ is the product $S^1 \times N_\Gamma$. In fact, we have the commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \pi_1(N_\Gamma) & \longrightarrow & \pi_1(N) & \longrightarrow & \Gamma \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & H_1(N_\Gamma) & \longrightarrow & H_1(N) & \longrightarrow & T \\
 & & \uparrow & & \uparrow & \nearrow \cong & \\
 \text{Tor}(H_1(N_\Gamma)) & \longrightarrow & \text{Tor}(H_1(N)) & & & &
 \end{array}$$

It follows from the commutativity, and from the exactness of the top horizontal sequence that the map $\pi_* : \text{Tor}(H_1(N_\Gamma)) \rightarrow \text{Tor}(H_1(N))$, and hence $(\pi_*)^* : \text{Ext}(H_1(N), \mathbb{Z}) \rightarrow \text{Ext}(H_1(N_\Gamma), \mathbb{Z})$ is trivial. It now follows from the naturality of the universal coefficient short exact sequence that the Euler class $e_\Gamma = \pi^*e$ of the fibration $p_\Gamma : M_\Gamma \rightarrow N_\Gamma$ is zero, i.e. M_Γ is the product $S^1 \times N_\Gamma$.

As (M, ω) is symplectic, the manifold $S^1 \times N_\Gamma$ inherits a symplectic structure $\omega_\Gamma = \pi^*\omega$ which satisfies $(p_\Gamma)_*[\omega_\Gamma] = \phi_\Gamma \in H^1(N_\Gamma)$ hence, by [FV06a, Proposition 4.4], $\Delta_{N_\Gamma, \phi_\Gamma}$ must be monic. However, it is immediate to see that the epimorphism $\gamma : \pi_1(N) \rightarrow \Gamma$ satisfies, just like α , the condition $\gamma(\pi_1(\Sigma)) \subsetneq \gamma(\pi_1(N \setminus \nu\Sigma))$, hence the corresponding twisted Alexander polynomial $\Delta_{N, \phi}^\gamma = \Delta_{N_\Gamma, \phi_\Gamma}$ vanishes, in contrast with our hypothesis.

For what concerns (2) and (3), when we apply the above construction with $G = \{e\}$, we get a manifold $S^1 \times N_\Gamma$ for which, respectively, N_Γ is a graph manifold or the canonical class is trivial. By [FV06b] the pair (N_Γ, ϕ_Γ) is fibered, which implies that so is (N, ϕ) . \square

Corollary 5.2. *Let M be a manifold endowed with a free circle action with torsion Euler class. Then M admits a symplectic structure with trivial canonical class if and only if it is a T^2 -bundle over T^2 .*

Proof. If the Euler class is trivial, as by [FV06b] N fibers over S^1 with torus fiber, $M = S^1 \times N$ is obviously a T^2 -bundle over T^2 . If the Euler class is non-trivial, keeping the notation in the proof of Theorem 5.1, M is the quotient of the action of the (abelian) deck transformation group Γ on the product manifold $M_\Gamma = S^1 \times N_\Gamma$, a T^2 -bundle over T^2 . By construction, this action covers the Γ -action on N_Γ whose quotient is N . The inverse image w.r.t. the projection $\pi : N_\Gamma \rightarrow N$ of a fiber Σ of the torus fibration of N determined by $\phi \in H^1(N)$ is a union of $|\Gamma/\gamma(\pi_1(\Sigma))| = 1$ fibers of the fibration of N_Γ (where the isomorphism $\gamma(\pi_1(\Sigma)) \cong \Gamma$ follows from the fact that the torsion part of $H_1(N)$ comes from the invariant homology of the fiber, although we do not actually need this fact). This means that the group Γ acts through bundle automorphisms with respect to the torus fibration $N_\Gamma \rightarrow S^1$, hence the lift of this action to M_Γ acts through bundle automorphisms with respect to the torus fibration $M_\Gamma \rightarrow T^2$. As tori quotients are tori, M is itself a T^2 -bundle over T^2 . \square

As for the case of Theorem 4.1, [MS96] guarantees that the manifolds above are the only symplectic manifolds with Kodaira dimension zero.

6. EXAMPLES

In this section we will analyze some interesting examples of 4-manifolds admitting a circle action with nontorsion Euler class, and discuss how the results of the previous sections allow us to determine whether they admit symplectic structures or not. Note that the conclusions of this section are not conditional to subgroup separability assumptions, as they follow from results as Theorem 3.4 and Theorem 4.1 that hold without such assumptions.

For technical reasons, in this section, given a homomorphism $\alpha : \pi_1(N) \rightarrow G$ to a finite group G , it will be convenient to use the multivariable (twisted) Alexander polynomial of N which is denoted by $\Delta_N^\alpha \in \mathbb{Z}[H]$, where H is the maximal free abelian quotient of $\pi_1(N)$. Its relation with the 1-variable polynomial $\Delta_{N,\phi}^\alpha \in \mathbb{Z}[t^{\pm 1}]$ (essentially given by specialization) is discussed in [FV06a, Section 3].

Consider the 3-torus T^3 and let C be a fiber of a fibration $T^3 \rightarrow T^2$, endowed with the framing induced by the fibration. Pick a meridian μ_C and the longitude λ_C determined by the framing. Next, let $K \subset S^3$ be an oriented knot. We denote by μ_K and λ_K its meridian and longitude. Now, splice the two exteriors to form the 3-manifold

$$(13) \quad T_K^3 = (T^3 \setminus \nu C) \cup (S^3 \setminus \nu K)$$

where the gluing map on the boundary 2-tori identifies μ_K with λ_C and λ_K with μ_C . Note that, if N_K is the 0-surgery of S^3 along K , and if m is the image of the canonically framed curve μ_K in N_K , then we can write T_K^3 as normal connected sum of T^3 and N_K , i.e. $T_K^3 = T^3 \#_{C=m} N_K$.

As the surgery of Equation (13) amounts to the substitution of a solid torus with a homology solid torus, respecting the boundary maps, and as the class of C is primitive, it is easy to see from the Mayer–Vietoris sequence that the inclusion maps induce isomorphisms $H_1(T^3) \xleftarrow{\cong} H_1(T^3 \setminus \nu C) \xrightarrow{\cong} H_1(T_K^3)$ which we use to identify these groups for the remainder of this section. Also, we pick a basis x, y, z for $H_1(T^3) \cong H_1(T_K^3)$ such that $z = [C]$.

Given $e \in H^2(T_K^3)$ we denote by $M_K(e)$ the total space of the S^1 -bundle over T_K^3 with Euler class e . Our first result covers two classes of examples of 4-manifolds M with $b_2^+(M) > 1$ that satisfy $SW_M = 1 \in \mathbb{Z}[H^2(M)]$, but do not admit symplectic structures (that would necessarily have $K = 0$).

Proposition 6.1. (1) *Let $K \subset S^3$ be a non-trivial knot with $\Delta_K = 1$; then, for all $e \neq 0 \in H^2(T_K^3)$ the manifold $M_K(e)$ satisfies $SW_{M_K(e)} = 1 \in \mathbb{Z}[H^2(M_K(e))]$ and admits no symplectic structures.*
 (2) *Let $K \subset S^3$ be any non-trivial knot; then, for $e = \pm PD(z)$ or $\pm 2PD(z) \in H^2(T_K^3)$, the manifold $M_K(e)$ satisfies $SW_{M_K(e)} = 1 \in \mathbb{Z}[H^2(M_K(e))]$ and admits no symplectic structures.*

Proof. As $b_1(T^3) \geq 3$, to compute $SW_{M_K(e)}$ we will use Theorem 3.2 and Equation (9). Using the injection $H_1(N_K) \xleftarrow{\cong} H_1(S^3 \setminus K) \rightarrow H_1(T_K^3)$ we view $\Delta_{N_K} \in \mathbb{Z}[H_1(N_K)]$ as an element in $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$. With this convention we can write the Alexander polynomial of T_K^3 in terms of the Alexander polynomials of T^3 and N_K . Indeed, using the product formula for Milnor torsion (see e.g. [Tu02]), and the fact that $\Delta_{T^3} = 1$ we get

$$(14) \quad \Delta_{T_K^3}(x, y, z) = \Delta_{T^3}(x, y, z) \cdot \Delta_{N_K}(z) = \Delta_K(z) \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}].$$

Using this equation, we obtain the statement on the Seiberg-Witten invariants: indeed, for the first class of manifolds, the result follows immediately from Equation (7), while for the second class it is a consequence of the choice of e , which implies that $0 \in H^2(M_K(e))$ is the only basic class with coefficient given by the sum of coefficients of Δ_K , namely 1 for all choices of K .

Note that in both cases the trivial class is the only basic class, in particular if $M_K(e)$ were symplectic its canonical would be trivial. However, it is easy to see that T_K^3 (that is irreducible, as it is obtained by gluing two irreducible manifolds along incompressible boundaries) does not satisfy the constraints determined in the proof of Theorem 4.1, namely does not fiber over S^1 . In fact, it is certainly not a torus bundle over S^1 : indeed, $T_K^3(e)$ contains a minimal genus surface of genus $g(K) + 1 > 1$ obtained by gluing a minimal genus Seifert surface of K with a section of the fibration $T^3 \setminus \nu C \rightarrow T^2 \setminus \nu\{pt\}$, hence its Thurston norm does not vanish, in contrast with torus bundles. \square

The third class of examples is somewhat more sophisticated than the previous ones, and uses implicitly the results of [LN91].

Proposition 6.2. *Let $K \subset S^3$ be a knot of genus 1 and $e \neq 0 \in H^2(T_K^3)$. The manifold $M_K(e)$ admits a symplectic structure if and only if K is fibered and e is not a multiple of $PD([C])$.*

Proof. Recall that we write $z = [C]$. First assume that K is fibered. Let $a \neq 0 \in H^2(T_K^3)$, note that

$$\ker(\cdot a) := \ker(H^1(N) \xrightarrow{\cup a} H^3(N) \xrightarrow{\cap [N]} \mathbb{Z}) = \{\phi \in H^1(N) \mid \phi(PD(a)) = 0\}$$

is a rank 2 subspace of $H^1(N)$. Furthermore, by Poincaré duality the ray $\mathbb{Z}a$ is determined by $\ker(\cdot a)$. In particular, if e is not a multiple of $PD(z)$ then

$$\{\phi \mid \phi \cup e = 0\} = \ker(\cdot e) \not\subset \ker(\cdot PD(z)) = \{\phi \mid \phi(z) = 0\}.$$

It follows that there exists $\phi \in H^1(T^3) = H^1(T^3 \setminus \nu C) = H^1(T_K^3)$ such that $\phi(z) \neq 0$ and $\phi \cup e = 0$. Using e.g. [McT99, Theorem 2.3] we see that $(T^3 \setminus \nu C, \phi)$ fibers over S^1 , and the fibration extends to T_K^3 via the fibration of $S^3 \setminus \nu K$ (while classes with $\phi(z) = 0$ have trivial Thurston norm and cannot be fibered). The “if” part now follows from Proposition 1.

We now turn to the proof of the “only if” statement. Observe that by obstruction theory we can define a proper map $T^3 \setminus \nu C \rightarrow S^1 \times D^2$ which realizes the class dual to z on cohomology and which extends the diffeomorphism $\partial(T^3 \setminus \nu C) \cong S^1 \times \partial D^2$ given by identifying λ_C and μ_C with S^1 and ∂D^2 respectively. Out of this we construct a map

$$T_K^3 = (T^3 \setminus \nu C) \cup (S^3 \setminus \nu K) \rightarrow (S^1 \times D^2) \cup (S^3 \setminus \nu K) = N_K$$

that, since K is non-trivial, induces the short exact sequence

$$1 \rightarrow \mathbb{Z} * \mathbb{Z} \rightarrow \pi_1(T_K^3) \rightarrow \pi_1(N_K) \rightarrow 1.$$

Given an epimorphism $\alpha : \pi_1(N_K) \rightarrow G$ onto a finite group we obtain a corresponding epimorphism of $\pi_1(T_K^3)$ onto G , that we denote for simplicity by α as well. We now consider the $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ -modules $H_1(T_K^3; \mathbb{Z}[G][x^{\pm 1}, y^{\pm 1}, z^{\pm 1}])$ and $H_1(N_K; \mathbb{Z}[G][z^{\pm 1}])$, where we view the latter as a $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ -module by considering $\mathbb{Z}[z^{\pm 1}]$ as the maximal quotient

of $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ invariant under the action of $\mathbb{Z} * \mathbb{Z}$ determined by the map $\mathbb{Z} * \mathbb{Z} \rightarrow \pi_1(T_K^3) \rightarrow H_1(T_K^3)$, whose image is exactly the subgroup generated by x and y . It now follows from the 5-terms exact sequence for group homology with coefficients (cf. [Br94, Section VII.6]) that we have a quotient map of $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ -modules

$$H_1(T_K^3; \mathbb{Z}[G][x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]) \rightarrow H_1(N_K; \mathbb{Z}[G][z^{\pm 1}]) \rightarrow 0.$$

This implies (cf. e.g. [Le67, Lemma 5, p. 76]) that there exists a polynomial $p^\alpha(x, y, z) \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ such that

$$\Delta_{T_K^3}^\alpha(x, y, z) = p^\alpha(x, y, z) \cdot \Delta_{N_K}^\alpha(z) \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}].$$

If K is not a fibered knot, it follows as in the proof of Corollary 3.6 that there exists an epimorphism $\alpha : \pi_1(N_K) \rightarrow G$ s.t. $\Delta_{N_K}^\alpha(z) = 0$ and hence $\Delta_{T_K^3}^\alpha = 0$. This implies, by Theorem 3.4, that $M_K(e)$ does not admit symplectic structures.

It remains to show that, even if K is fibered, $M_K(e)$ cannot be symplectic if $e = \lambda PD(z)$. (Observe that some of these cases have already been excluded in Proposition 6.1.) For all those cases, as $p^*PD(z)$ is torsion, Theorem 3.2 shows that the only 4-dimensional basic classes are torsion. From this, it is immediate to see that the set of basic classes will either violate Equation (11) or, when that is satisfied (i.e. when $SW_{M_K(e)} = 1 \in \mathbb{Z}[H^2(M_K(e))]$, which happens if $|\lambda| = 1$ or 2 , or if $\Delta_K = 1$) then T_K^3 violates again Theorem 4.1, as in the proof of Proposition 6.1. \square

Remark. The proof of Proposition 6.2 can easily be generalized as follows: let Y be any closed 3-manifold, $C \subset Y$ a closed framed curve such that $[C]$ is a primitive element of $H_1(Y)$ and K any non-fibered genus one knot, then $M_K(e)$ is not symplectic for any $e \in H^2(Y_K)$. Here Y_K is defined analogously to T_K^3 , i.e. $Y_K = Y \setminus \nu C \cup S^3 \setminus \nu K$.

7. CONSTRUCTION OF SYMPLECTIC FORMS

For the convenience of the reader we recall the statement of Theorem 4. The subsequent sections will provide the details of its proof.

Theorem 7.1. *Let M be a 4-manifold admitting a free circle action. Let $\psi \in H^2(M; \mathbb{R})$ such that $\psi^2 > 0 \in H^4(M; \mathbb{R})$ and such that $p_*(\psi) \in H^1(N; \mathbb{R})$ can be represented by a non-degenerate closed 1-form. Then there exists an S^1 -invariant symplectic form ω on M with $[\omega] = \psi \in H^2(M; \mathbb{R})$.*

Combining Theorem 4 with the results of the previous sections, we will obtain Theorem 5, that characterizes completely the symplectic cone for some classes of manifolds, and whose statement we recall here for sake of clarity.

Theorem 7.2. *Let M be a 4-manifold with free S^1 -action such that the orbit space N is a graph manifold or has vanishing Thurston norm. Then a class $\psi \in H^2(M; \mathbb{R})$ can be represented by a symplectic form if and only if $\psi^2 > 0$ and $p_*(\psi) \in H^1(N; \mathbb{R})$ lies in the open cone on a fibered face of the unit ball of the Thurston. Furthermore, we can assume that the symplectic form is S^1 -invariant.*

In order to prove this theorem, in view of Theorem 4 and the results of the previous sections we just need to show that whenever $\psi \in H^2(M; \mathbb{R})$ can be represented by a symplectic

form, the class $p_*(\psi) \in H^1(N; \mathbb{R})$ lies in the open cone on a top-dimensional face of the unit ball of the Thurston norm. This property is quite likely to hold in generality (and easy to prove for the case that $M = S^1 \times N$). However, to provide a proof for the case at hand we will have to assume some extra conditions, satisfied in our case, that are summarized in the next proposition.

Proposition 7.3. *Let M be a 4-manifold with a free circle action. Denote by $p : M \rightarrow N$ the projection map to the orbit space. Assume that for any symplectic form ω with $p_*([\omega]) \in H^1(N)$ primitive the pair $(N, p_*([\omega]))$ fibers over S^1 . Then for any symplectic form ω the class $p_*([\omega]) \in H^1(N; \mathbb{R})$ can be represented by a non-degenerate closed 1-form.*

Proof. First let ω be a symplectic form such that $p_*([\omega]) \in H^1(N; \mathbb{Q})$. The argument in Section 3 shows that $p_*([\omega]) \neq 0$. Therefore we can find $s \in \mathbb{Q}$ such that $sp_*([\omega]) = p_*([s\omega])$ is a primitive element in $H^1(N)$. By assumption $(N, sp_*([\omega]))$ fibers over S^1 , in particular $sp_*([\omega])$ (and hence $p_*([\omega])$) can be represented by a non-degenerate closed 1-form.

Now let ω be a symplectic form such that $p_*([\omega]) \in H^1(N; \mathbb{R}) \setminus H^1(N; \mathbb{Q})$, and let C be the open cone over the face of the unit ball of the Thurston norm in which C lies. Since the vertices of the Thurston norm ball are rational (cf. [Th86, Section 2]), and by the openness of the symplectic condition, we can find a symplectic form ω' on M such that $p_*([\omega'])$ is in $H^1(N; \mathbb{Q})$ and is contained in the cone C as well. By the previous observation it follows that there exist at least one element of C (namely $p_*([\omega'])$ itself) that can be represented by a non-degenerate closed 1-form. But then by [Th86, Theorem 5] all its elements, in particular $p_*([\omega])$, can be represented by non-degenerate closed 1-forms. \square

Proof of Theorem 7.2. By Corollary 3.6, Proposition 3.7, Theorem 5.1 and the results of [FV06b] we know that for any symplectic form ω with $p_*([\omega]) \in H^1(N)$ primitive the pair $(N, p_*([\omega]))$ fibers over S^1 . Proposition 7.3 then asserts that for any symplectic form ω the class $p_*([\omega]) \in H^1(N; \mathbb{R})$ can be represented by a non-degenerate closed 1-form. This, together with Theorem 4, proves the corollary as stated. \square

Let W be a 4-manifold. The set of all elements of $H^2(W; \mathbb{R})$ which can be represented by a symplectic form is called the *symplectic cone* of W . Note that the symplectic cone is closed under multiplication by a scalar. Theorem 7.2 lets us determine the symplectic cone for a significant class of 4-manifolds. This suggests that the symplectic cone shares the properties of the fibered cone of a 3-manifold. More precisely we propose the following conjecture.

Conjecture 7.4. *Let W be a symplectic 4-manifold. Then there exists a (possibly non-compact) polytope $C \subset H^2(W; \mathbb{R})$ with the following properties:*

- (1) *The dual polytope in $H_2(W; \mathbb{R})$ is compact, symmetric, convex and integral.*
- (2) *There exist open top-dimensional faces F_1, \dots, F_s of C such that the symplectic cone coincides with all non-degenerate elements in the cone on F_1, \dots, F_s .*

It follows immediately from Theorem 7.2 and the results of [Th86] that the conjecture holds for W a circle bundle over a graph manifold or a manifold with vanishing Thurston norm.

7.1. Outline of the proof of Theorem 7.1. In this section we will give a proof of Theorem 7.1 modulo some technical lemmas which will be proved in Sections 7.2, 7.3 and 7.4.

For the remainder of this section let M be a 4-manifold admitting a free circle action. Let $\psi \in H^2(M; \mathbb{R})$ such that $\psi^2 > 0 \in H^4(M; \mathbb{R})$ and such that $p_*(\psi) \in H^1(N; \mathbb{R})$ can be represented by a non-degenerate closed 1-form α .

Lemma 7.5. *There exists a 1-form β on N such that $\alpha \wedge \beta$ is closed and $[\beta \wedge \alpha] = e \in H^2(N; \mathbb{R})$.*

In the case that $p_*(\psi)$ is integral this lemma is stated in [FGM91, Lemma 15]. We give the proof of Lemma 7.5 in Section 7.3.

Now let $\gamma = \beta \wedge \alpha$. Since $[\gamma] = e \in H^2(N; \mathbb{R})$ we can find a 1-form η (namely a connection 1-form for $M \rightarrow N$) on M with the following properties:

- (1) η is invariant under the S^1 -action,
- (2) the integral of η over a fiber equals 1, and
- (3) $d\eta = p^*(\gamma)$.

We refer to [Ni00] and [Ro98] for more details. Note that (1) and (2) imply that η is non-trivial on any non-trivial vector tangent to a fiber.

Note that $d(p^*(\alpha) \wedge \eta) = p^*(\alpha \wedge \gamma) = p^*(\alpha \wedge \alpha \wedge \beta) = 0$. We can therefore consider $[\psi] - [p^*(\alpha) \wedge \eta] \in H^2(M; \mathbb{R})$. It follows easily from $p_*([\psi]) = [\alpha]$ and the second property of η that $p_*([\psi] - [p^*(\alpha) \wedge \eta]) = 0 \in H^1(N; \mathbb{R})$. By the exact sequence (1) we can therefore find $h \in H^2(N; \mathbb{R})$ with $p^*(h) = [\psi] - [p^*(\alpha) \wedge \eta]$. By assumption we have $\psi^2 \neq 0$, it follows now easily from the discussion in Section 2 that $p^*(h) \cup [p^*(\alpha) \wedge \eta] \neq 0 \in H^4(M; \mathbb{R})$. By the Gysin sequence the map $p_* : H^4(M; \mathbb{R}) \rightarrow H^3(N; \mathbb{R})$ is an isomorphism, we therefore also get

$$h \cup [\alpha] = p_*(p^*(h) \cup [p^*(\alpha) \wedge \eta]) \neq 0 \in H^3(N; \mathbb{R}).$$

We will prove the following lemma in Section 7.4.

Lemma 7.6. *Given $h \in H^2(N; \mathbb{R})$ with $h \cup [\alpha] \neq 0$ we can find a representative Ω of h such that $\Omega \wedge \alpha \neq 0$ everywhere.*

It is now clear that the following claim concludes the proof of Theorem 4.

Claim.

$$\omega = p^*(\Omega) + p^*(\alpha) \wedge \eta$$

is an S^1 -invariant symplectic form on M .

It is clear that ω is S^1 -invariant. We compute

$$d\omega = d(p^*(\Omega) + p^*(\alpha) \wedge \eta) = p^*(\alpha) \wedge d\eta = p^*(\alpha \wedge \gamma) = p^*(\alpha \wedge \alpha \wedge \beta) = 0.$$

It remains to show that $\omega \wedge \omega$ is non-zero everywhere. For any point in M pick a basis a, b, c, d for the tangent space such that:

- (1) $p_*(a), p_*(b)$ are a basis for the tangent space of a leaf of the foliation, put differently, $\alpha(p_*(a)) = \alpha(p_*(b)) = 0$ and $p_*(a), p_*(b)$ are linearly independent,
- (2) c is tangent to the fibers of the S^1 -fibration $M \rightarrow N$,
- (3) $\alpha(p_*(d)) \neq 0$.

Note that $p_*(c) = 0$ and $p^*(\alpha)$ vanishes on a, b, c . It is now easy to see that

$$(\omega \wedge \omega)(a, b, c, d) = 2p^*(\Omega)(a, b) \cdot p^*(\alpha)(d) \cdot \eta(c) = 2\Omega(p_*(a), p_*(b)) \cdot \alpha(p_*(d)) \cdot \eta(c),$$

which is clearly non-zero. This concludes the proof of the claim and hence the proof of Theorem 7.1

7.2. Non-degenerate closed 1-forms and dual curves. Before we can prove Lemmas 7.5 and 7.6 we need two lemmas regarding non-degenerate closed 1-forms and curves representing homology classes. Throughout this section α will be a non-degenerate closed 1-form on N . Note that α defines a foliation which we denote by \mathcal{F} . In this section we prove the following two lemmas.

Lemma 7.7. *Let α be a non-degenerate closed 1-form on N with corresponding foliation \mathcal{F} and let $p \in N$. For every $h \in H^2(N; \mathbb{Z})$ with $h \cup [\alpha] = 0$ there exists a smoothly embedded closed (possibly disconnected) curve c with $PD([c]) = h$ which lies in a leaf of \mathcal{F} and goes through p .*

Lemma 7.8. *Let α be a non-degenerate closed 1-form on N with corresponding foliation \mathcal{F} . For every $h \in H^2(N; \mathbb{Z})$ with $h \cup [\alpha] \neq 0$ and for every $p \in N$ there exists a connected, smoothly embedded curve with $PD([c]) = h$ that is transverse to \mathcal{F} and which goes through p .*

A proof for Lemma 7.7 is given in [Lt87, Proposition II.2], but we did not find a reference for Lemma 7.8, even though presumably it is well-known. For completeness' sake we include the proofs of both lemmas. The proofs will require the remainder of this section.

Let α be a non-degenerate closed 1-form on N with corresponding foliation \mathcal{F} . We first pick a metric g on N . We let v' be the unique vector field on N with the property that for any $p \in N$ and any $w \in T_p N$ we have $g(v'(p), w) = \alpha(w)$. Note that this implies that $\alpha(v'(p)) \neq 0$ for all p . We then define a new vector field v by

$$v(p) = \frac{v'(p)}{\alpha(v'(p))}.$$

Note that $\alpha(v(p)) = 1$ for all $p \in N$. We denote by $F : N \times \mathbb{R} \rightarrow N$ the flow corresponding to $-v$, i.e. for any $p \in N, s \in \mathbb{R}$ we have

$$\frac{\partial}{\partial t} F(p, t)|_{t=s} = -v(F(p, s)).$$

In the following we identify S^1 with $[0, 1]/0 \sim 1$. Given a curve c we define $\Phi_c(t) = \int_{c|_{[0, t]}} \alpha$, where $c|_{[0, t]}$ denotes the restriction of the map $c : [0, 1] \rightarrow N$ to the interval $[0, t]$.

Proof of Lemma 7.7. Let α be a non-degenerate closed 1-form on N and $h \in H^2(N; \mathbb{Z})$ with $h \cup [\alpha] = 0$. Let d be any smoothly embedded connected curve dual to h with $d(0) = p$. Note that $\Phi_d(1) = \int_d \alpha = h \cup [\alpha] = 0$.

We consider the following homotopy

$$\begin{aligned} H : [0, 1] \times [0, 1] &\rightarrow N \\ (t, s) &\mapsto F(d(t), s\Phi_d(t)). \end{aligned}$$

This is clearly a smooth map. Since $\Phi_d(1) = 0$ this descends in fact to a homotopy $H : S^1 \times [0, 1] \rightarrow N$. Note that $H(t, 0) = d(t)$ for all t . We now consider the closed curve d'

defined by $d'(t) = H(t, 1)$. This curve is smooth, but it has possibly self-intersections. Note that $d'(0) = p$. We claim that d' lies in a leaf of \mathcal{F} , i.e. we claim that $\Phi_{d'}(s) = 0$ for all $s \in [0, 1]$. So let $s \in [0, 1]$. Note that the homotopy H can be used to show that the curve $d'|_{[0, s]}$ is homotopic to the following curve

$$\begin{aligned} d'' : [0, s] &\rightarrow N \\ t &\mapsto \begin{cases} d(2t), & \text{for } t \in [0, s/2] \\ F(d(s), (2t - s)\Phi_d(t)), & \text{for } t \in [s/2, s]. \end{cases} \end{aligned}$$

Since α is a closed form we see that $\Phi_{d'}(s) = \Phi_{d''}(s) = 0$.

We get the required curve c by removing self-intersections of the curve d' in the leaf of \mathcal{F} . Note that the smoothing can increase the number of components of the curve. \square

Proof of Lemma 7.8. Let α be a non-degenerate closed 1-form on N and $h \in H^2(N; \mathbb{Z})$ with $h \cup [\alpha] = 0$. Let d be any smoothly embedded connected curve dual to h with $d(0) = p$. Note that $\Phi_d(1) = \int_d \alpha = h \cup [\alpha] \neq 0$. Let $\Psi : [0, 1] \rightarrow \mathbb{R}$ be a function with $\Psi(1) = \Phi_d(1)$ such that $\Psi'(t) > 0$ (respectively $\Psi'(t) < 0$) for all t .

We consider the following homotopy

$$\begin{aligned} H : [0, 1] \times [0, 1] &\rightarrow N \\ (t, s) &\mapsto F(d(t), s(\Phi_d(t) - \Psi(t))). \end{aligned}$$

This is clearly a smooth map. Since $\Phi_d(1) = \Psi(1)$ this descends in fact to a homotopy $H : S^1 \times [0, 1] \rightarrow N$.

Note that $H(t, 0) = d(t)$ for all t . We now consider the curve d' defined by $d'(t) = H(t, 1)$. Note that $d'(0) = p$. The argument of the proof of Lemma 7.7 shows that $\Phi_{d'}(t) = \Psi(t)$. Since $\Psi'(t) \neq 0$ for all t it follows that d' is transversal to \mathcal{F} . Furthermore note that $d : [0, 1]/0 \sim 1 \rightarrow N$ is smooth everywhere except possibly at $t = 0$. Using a homotopy we can turn d' into a transversal smoothly embedded curve c dual to h which goes through p . \square

7.3. Proof of Lemma 7.5. We are now ready to prove the first of the two auxiliary lemmas, i.e. we will prove the following claim.

Claim. Let α be a non-degenerate closed 1-form on N and $e \in H^2(N; \mathbb{Z})$ such that $e \cup [\alpha] = 0$. There exists a 1-form β on N such that $\alpha \wedge \beta$ is closed and $[\beta \wedge \alpha] = e \in H^2(N; \mathbb{R})$.

By Lemma 7.7 we can find an oriented smoothly embedded curve c dual to $e \in H^2(N; \mathbb{Z})$ such that $\alpha|_c \equiv 0$. We denote the components of c by c_1, \dots, c_m . We now consider $S^1 \times D^2$ with the coordinates $e^{2\pi it}, x, y$ and we orient $S^1 \times D^2$ by picking the equivalence class of the basis $\{\partial_x, \partial_y, \partial_t\}$.

Using the orientability of the N and of the leaves of the foliation we use a standard argument to show that for $i = 1, \dots, m$ we can pick a map

$$f_i : S^1 \times D^2 \rightarrow N$$

with the following properties:

- (1) f_i is an orientation preserving diffeomorphism onto its image.
- (2) f_i restricted to $S^1 \times 0$ is an orientation preserving diffeomorphism onto c_i .
- (3) $\alpha((f_i)_*(\partial_t)) = 0$.

$$(4) \quad \alpha((f_i)_*(\partial_x)) = 0.$$

$$(5) \quad \text{There exists an } r_i \in (0, \infty) \text{ such that } \alpha((f_i)_*(\partial_y)) = r_i \text{ everywhere.}$$

Note that (3), (4) and (5) are equivalent to $f_i^*(\alpha) = r_i \cdot dy$.

For $i = 1, \dots, m$ we now pick a function $\rho_i : D^2 \rightarrow \mathbb{R}_{\geq 0}$ such that the closure of the support of ρ lies in the interior of D^2 and such that $\int_{D^2} \rho_i = \frac{1}{r_i}$. We define the following 1-form on $S^1 \times D^2$:

$$\beta'_i(z, x, y) = \rho_i(x, y) \cdot dx.$$

Note that

$$(15) \quad d(\beta'_i \wedge f_i^*(\alpha)) = d(\beta'_i \wedge r_i \cdot dy) = d(r_i \rho_i(x, y) \cdot dx \wedge dy) = 0.$$

Furthermore for any $z \in S^1$ we have

$$(16) \quad \int_{z \times D^2} \beta'_i \wedge f_i^*(\alpha) = \int_{z \times D^2} r_i \rho_i(x, y) \cdot dx \wedge dy = 1.$$

For $i = 1, \dots, m$ we now define the following 1-form on N :

$$\beta_i(p) = \begin{cases} 0, & \text{if } p \in N \setminus f_i(S^1 \times D^2) \\ (f_i^{-1})^*(\beta'_i(q)), & \text{if } p = f_i(q) \text{ for some } q \in S^1 \times D^2. \end{cases}$$

Furthermore we let $\beta = \sum_{i=1}^m \beta_i$. We claim that β has all the required properties.

First note that β is C^∞ by our condition on the support of ρ_i . Furthermore it follows immediately from (15) that $\beta \wedge \alpha$ is closed. Finally we have to show that $\beta \wedge \alpha$ represents e .

In order to show that $\beta \wedge \alpha$ represents e in $H^2(N; \mathbb{R}) = \text{Hom}(H_2(N; \mathbb{Z}), \mathbb{R})$ it is enough to show that for any embedded oriented surface $S \subset N$, we have

$$\int_S \beta \wedge \alpha = e([S]).$$

We first note that $e([S]) = c \cdot s$. It is therefore enough to show that for any embedded oriented surface $S \subset N$, we have

$$\int_S \beta_i \wedge \alpha = c_i \cdot S.$$

In fact, given such a surface we can isotope S in such a way that S intersects the curve c ‘vertically’, i.e. we can assume that

$$f_i(S^1 \times D^2) \cap S = \epsilon_1 \cdot f_i(z_1 \times D^2), \dots, \epsilon_k \cdot f_i(z_k \times D^2)$$

for disjoint z_i and $\epsilon_i \in \{-1, 1\}$. We view this equality as an equality of oriented manifolds, where we give $z_i \times D^2$ the orientation given by the basis $\{\partial_x, \partial_y\}$. In particular S is transverse to c_i . In this case we have

$$c_i \cdot S = \sum_{j=1}^k \epsilon_j.$$

On the other hand it follows from (16) that

$$\int_S \beta_i \wedge \alpha = \sum_{j=1}^k \int_{\epsilon_j \cdot (z_j \times D^2)} f_i^*(\beta_i) \wedge f_i^*(\alpha) = \sum_{j=1}^k \int_{\epsilon_j \cdot (z_j \times D^2)} \beta'_i \wedge f_i^*(\alpha) = \sum_{j=1}^k \epsilon_j.$$

This concludes the proof that β has all the required properties.

7.4. Proof of Lemma 7.6. The following claim is the last missing piece in the proof of Theorem 4.

Claim. Let α be a non-degenerate closed 1-form on N . Given $h \in H^2(N; \mathbb{R})$ with $h \cup [\alpha] \neq 0$ we can find a representative Ω of h such that $\Omega \wedge \alpha \neq 0$ everywhere.

We first consider the case that h is represented by an integral class, i.e. by an element in the image of the map $H^2(N; \mathbb{Z}) \rightarrow H^2(N; \mathbb{R})$. Let \mathcal{F} be the foliation corresponding to α .

Now let p_1, p_2, \dots a sequence of points in N which is dense everywhere. Using Lemma 7.8 we can pick for each i a curve c_i transverse to \mathcal{F} which goes through p_i and which represents h . Since N is orientable we can pick maps

$$f_i : S^1 \times D^2 \rightarrow N$$

such that

- (1) f_i is an orientation preserving diffeomorphism onto its image (where we again view $S^1 \times D^2$ with the orientation given by $\{\partial_x, \partial_y, \partial_t\}$).
- (2) f_i restricted to $S^1 \times 0$ is an orientation preserving diffeomorphism onto c_i .
- (3) $\alpha((f_i)_*(\partial_x)) = 0$.
- (4) $\alpha((f_i)_*(\partial_y)) = 0$.
- (5) $\alpha((f_i)_*(\partial_t)) > 0$.

Note that (3) and (4) are equivalent to saying that $(f_i)_*(\partial_x)$ and $(f_i)_*(\partial_y)$ are tangent to the leaves of the foliation \mathcal{F} . Also note that on $S^1 \times D^2$ we have $dx \wedge dy \wedge (f_i)^*(\alpha) \neq 0$.

By compactness we can find i_1, \dots, i_k such that

$$(17) \quad \bigcup_{j=1}^k f_{i_j}(S^1 \times \frac{1}{2}D^2) = N.$$

Without loss of generality we can assume that $i_j = j$.

Now we pick a function $\rho : D^2 \rightarrow \mathbb{R}_{\geq 0}$ such that the following conditions hold:

- (1) $\int_{D^2} \rho = \frac{1}{k}$,
- (2) ρ is strictly positive on $\frac{1}{2}D^2$, and
- (3) the closure of the support of ρ lies in the interior of D^2 .

Let Ω' be the 2-form on $S^1 \times D^2$ given by

$$\Omega'(z, x, y) = \rho(x, y) dx \wedge dy.$$

Clearly Ω' is closed and for any $z \in S^1$ we have $\int_{z \times D^2} \Omega' = \frac{1}{k}$. For $i = 1, \dots, k$ we now define the following 2-form on N :

$$\Omega_i(p) = \begin{cases} 0, & \text{if } p \in N \setminus f_i(S^1 \times D^2) \\ (f_i^{-1})^*(\Omega'(q)), & \text{if } p = f_i(q) \text{ for some } q \in S^1 \times D^2. \end{cases}$$

As in the proof of Lemma 7.5 we see that Ω_i is smooth, Ω_i is closed and $[\Omega_i] = \frac{1}{k}h \in H^2(N; \mathbb{R})$. Now let $\Omega(h) = \sum_{i=1}^k \Omega_i$. Clearly $[\Omega(h)] = h \in H^2(N; \mathbb{R})$, and it easily follows from (17) and all the other conditions that $\Omega(h) \wedge \alpha > 0$ everywhere.

We now turn to the general case, i.e. to the case $h \in H^2(N; \mathbb{R})$ is not necessarily integral.

Lemma 7.9. *Let $h \in H^2(N; \mathbb{R})$ with $h \cup [\alpha] > 0$. Then we can find $m \in \mathbb{N}$, integral h_1, \dots, h_m and $a_1, \dots, a_m \in \mathbb{R}_{\geq 0}$ such that $h_i \cup [\alpha] > 0$ for all i and such that $h = \sum_{i=1}^m a_i h_i$.*

We first show that Lemma 7.9 implies Lemma 7.6. Indeed, given $h \in H^2(N; \mathbb{R})$ with $h \cup [\alpha] > 0$ we pick integral h_1, \dots, h_m and $a_1, \dots, a_m \in \mathbb{R}_{\geq 0}$ as above. Then we define $\Omega(h_1), \dots, \Omega(h_m)$ as above. We let

$$\Omega = \sum_{i=1}^m a_i \Omega(h_i).$$

We see that

$$\Omega(h) \wedge \alpha = \sum_{i=1}^m a_i \Omega(h_i) \wedge \alpha > 0$$

everywhere. This concludes the proof of Lemma 7.6 assuming Lemma 7.9.

We now turn to the proof of Lemma 7.9. It is easy to see that we can pick a basis e_1, \dots, e_n for $H^1(N; \mathbb{Q})$ such that $e_i \cup [\alpha] > 0$ for all $i = 1, \dots, m$. We use this basis to identify $H^2(N; \mathbb{R})$ with \mathbb{R}^n . We say that $h \in H^2(N; \mathbb{R})$ with $h \cup [\alpha] > 0$ has property (*) if there exists $m \in \mathbb{N}$, integral h_1, \dots, h_m and $a_1, \dots, a_m \in \mathbb{R}_{\geq 0}$ such that $h_i \cup [\alpha] > 0$ for all i and such that $h = \sum_{i=1}^m a_i h_i$. Note that if h_1, h_2 have property (*), then $h_1 + h_2$ also has property (*).

We now define

$$P(m) = \max\{(*) \text{ holds for all } g = (g_1, \dots, g_n) \in H^2(N; \mathbb{R}) \text{ with } g_1, \dots, g_m \in \mathbb{Q}\}$$

Clearly we have to show that $P(0)$ holds. Note that $P(n)$ holds since any rational element of $H^2(N; \mathbb{R})$ is a non-negative multiple of an integral element.

We now show that $P(m+1)$ implies that $P(m)$ holds as well. So assume $P(m+1)$ holds and that we have

$$g = (g_1, \dots, g_m, g_{m+1}, \dots, g_n)$$

with $g_1, \dots, g_m \in \mathbb{Q}$ and $h \cdot [\alpha] > 0$. By continuity we can find $r > 0$ such that $g_{m+1} - r \in \mathbb{Q}$ and with the property that

$$(g_1, \dots, g_m, g_{m+1} - r, \dots, g_n) \cdot [\alpha] > 0.$$

We write

$$(g_1, \dots, g_m, g_{m+1}, \dots, g_n) = (g_1, \dots, g_m, g_{m+1} - r, \dots, g_n) + r e_{m+1}.$$

The claim now follows from $P(m+1)$ and $e_{m+1} \cup [\alpha] > 0$.

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